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A finite steps algorithm for solving convex feasibility problems

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Abstract This paper develops a new variant of the classical alternating projection method for solving convex feasibility problems where the constraints are given by the intersection of two convex cones in a Hilbert space. An extension to the feasibility problem for the intersection of two convex sets is presented as well. It is shown that one can solve such problems in a finite number of steps and an explicit upper bound for the required number of steps is obtained. As an application, we propose a new finite steps algorithm for linear programming with linear matrix inequality constraints. This solution is computed by solving a sequence of a matrix eigenvalue decompositions. Moreover, the proposed procedure takes advantage of the structure of the problem. In particular, it is well adapted for problems with several small size constraints.

Keywords Convex optimization · Linear matrix inequality · Eigenvalue problem · Alternating projections

1 Introduction

Convex optimization problems are theoretically well understood and there exists an array of qualitative and quantitative results for their solution. A challenging problem that often remains is to reduce the complexity of numerical solutions. In this paper we do not intend to address completely this issue but rather provide a new direction towards finiteness of the algorithm. Our main result is that one can solve some *convex optimization* problems in a *finite number of steps*. Moreover, the method can be easily

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implemented for many interesting problems. One must only carry out the numerical computation of the projection onto some associated cones.

As a starting point, consider the linear conic optimization problem

$$\begin{array}{ll} \text{minimize} & \langle c, x \rangle \\ \text{subject to} & x \in \mathcal{C}_1 \cap \mathcal{C}_2, \end{array} \tag{1}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are given convex cones in a Hilbert space, endowed with a scalar product $\langle ., . \rangle$.

Problem (1) represents an important class of convex optimization problems. Two classical examples of great interest are those of linear programming and linear optimization over constraints in cones of positive semidefinite matrices. We show that one can solve interesting classes of such problems (1) by a finite step algorithm.

Optimization over linear constraints in the cone of positive semidefinite matrices has a wide spread applicability, including e.g. several problems of interest in control theory (Boyd et al. 1994). Henceforth, a particular interest of this paper is to provide a procedure for solving such problems known as semidefinite programming (SDP) or linear matrix inequality (LMI) feasibility problem (Boyd et al. 1994, Nesterov and Nemerovski 1994, Vandenberghe and Boyd 1996). These kind of optimization problems possesses polynomial time complexity and have been earlier treated by interior point methods due to Karmarkar (1984). A general theory for interior point methods can be found in Nesterov and Nemerovski (1994). Note that the projective method proposed in Gahinet and Nemirovski (1997), for solving LMI feasibility problems converges in a finite number of steps. This method is related to the Dikin ellipsoid method, combined with iterative step size selections and successive projections onto a linear subspace. In this paper, we propose an approach that avoids the need for step-size selections and uses a simple projection scheme. A further advantage is that it can be used to solve general convex feasibility problems. Our approach for solving SDP problems is based on the method of alternating projections (MAP). The idea of using the MAP for solving systems of linear equalities is due to Von Neumann (1950). Generalizations of the MAP can be found in Bregman (1965) and Gubin et al. (1967). The MAP method was also used in Han (1988), to find the best approximate to any given point in a Hilbert space from the intersection of a finite collection of closed convex sets. For a survey on MAP algorithms we refer to Bauschke and Borwein (1996). Also, in Skelton et al. (1997), applications to control theory are discussed. We emphasize that the classical MAP method can only converge asymptotically to a solution (and thus is not computed within finite number of iterations). Here, by introducing a modification of the MAP method, we show that we can solve general convex feasibility problems using a finite steps algorithm. Also, specialized algorithms for LMI optimization are provided and the numerical solution can be computed by consecutively solving a system of linear equations and a symmetric eigenvalue problem. Moreover, the presented procedure can take advantage of the structure of the problem. In particular, it is numerically attractive for problems with several small size constraints. An application of our approach can be found in Orsi et al. (2003), where a comparison between our algorithm and convex optimization packages such as SeDuMi (which is one of the most powerful LMI solvers) proposed by Sturm (1999), has been made for solving linear matrix inequalities. It is shown in Orsi et al. (2003) that the proposed algorithm performs favorably on LMIs with a large number of constraints, relative to the number of variables.

One of the limitations of most convex optimization algorithms is that the constraints are assumed to have a non empty interior. When such condition fails, there may not exist any procedure that is guaranteed to find a solution. The exact complexity of the *semidefinite feasibility* problem (i.e., finding a positive semidefinite matrix in some affine space) is still unknown (Ramana 1997).

We notice that when applying the MAP to the semidefinite feasibility problem (with empty interior), usually only asymptotic convergence results can be obtained. Nevertheless, we show that it is still possible to use the proposed algorithm to check in finitely many steps the *infeasibility* of this problem.

The structure of this paper is as follows. In Sect. 2, we present our general approach to solve convex feasibility problems. First, the case of convex cones is treated and from that, by the conification procedure, the analysis is extended to the general convex case. Section 3 presents specific finite step algorithms for solving the strict LMI feasibility problems. Extensions to semidefinite LMI feasibility problems and semidefinite programming appear in Sect. 4. Conclusions are given in Sect. 5.

Notation. We use the following notation. $\mathcal{H}$ denotes a real Hilbert space equipped with a positive definite inner product $\langle \cdot, \cdot \rangle$. $\overset{o}{\mathcal{S}}$ denotes the interior of $\mathcal{S}$ in $\mathcal{H}$. $\bar{\mathcal{S}}$ denotes the closure of $\mathcal{S}$ in $\mathcal{H}$. $\mathbb{R}^{m\times n}$ denotes the set of real matrices of size $m \times n$. $\mathcal{S}_n$ denotes the set of real symmetric matrices of size n. $\mathcal{S}_n^+$ denotes the subset of positive semidefinite matrices of size n. M^T denotes the transpose of the matrix M. $\mathbf{Tr}(M)$ is the sum of diagonal elements of a square matrix M. I denotes the identity matrix, with size determined from the context. Given square matrices $M_1, \ldots, M_l$, the matrix $\mathbf{diag}(M_1, \ldots, M_l)$ denotes the block diagonal matrix with ith block M_i.

2 General approach

2.1 Preliminaries

The following is a classical result on the minimum distance function from a given point to a convex set (see, e.g., Luenberger 1969).

Theorem 2.1 *Let $\mathcal{C}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Then for any given element x in $\mathcal{H}$ there exists a unique element $\hat{x}$ in $\mathcal{C}$ such that*

$$\|x - \hat{x}\| \leq \|x - y\|, \quad \forall y \in \mathcal{C}. \tag{2}$$

Furthermore, a necessary and sufficient condition such that $\hat{x}$ satisfies (2) is given by

$$\langle x - \hat{x}, y - \hat{x} \rangle \leq 0, \quad \forall y \in \mathcal{C}. \tag{3}$$

The above result shows that the mapping $\hat{x} \triangleq \mathcal{P}_{\mathcal{C}}(x)$ is well defined. The operator $\mathcal{P}_{\mathcal{C}}$ is called the metric projection operator on $\mathcal{C}$. For some special convex sets, the condition (3) can be used to characterize explicitly $\mathcal{P}_{\mathcal{C}}$.

Now, assume that we have a finite collection $\mathcal{C}_1, \ldots, \mathcal{C}_N$ of closed convex subsets of a Hilbert space. The feasibility problem we are considering is to find an element in the intersetion of these given subsets. This problem can be solved by successive projections on these subsets. This method is due to Von Neumann (1950), for linear subspaces. Its generalization to arbitrary convex subsets can be found in Bregman (1965). Explicitly, we work with the following projection algorithm. Let $\mathcal{C}_1, \ldots, \mathcal{C}_N$ be closed convex sets in a Hilbert space $\mathcal{H}$, such that $\mathcal{C}_1 \cap \cdots \cap \mathcal{C}_N \neq \emptyset$. The *alternating projection algorithm* consists in the sequence (x_n), defined by the iteration scheme

$$x_{i+1} = \mathcal{P}_{\mathcal{C}_{\phi(i)}}(x_i), \qquad \text{where } \phi(i) = (i \bmod N) + 1. \tag{4}$$

The above projection algorithm usually requires an infinite numbers of iterations. Moreover, in an arbitrary Hilbert space it converges only weakly to an element of the intersection of $\mathcal{C}_1, \ldots, \mathcal{C}_N$. Strong convergence holds if the Hilbert space is finite-dimensional, or under other conditions. However, we will see that the situation improves when the convex sets are conic. In particular, we provide a modified algorithm that determines a feasible point in a finite number of steps and derive a bound on the number of necessary iteration steps. Before stating and proving our main technical result, we first recall some basic properties of convex cones.

2.2 Convex cones

Recall that a cone in a Hilbert space $\mathcal{H}$ is a subset invariant under nonnegative scalar multiplication. Here we will exhibit some interesting properties of a special class of cones called self dual.

Definition 2.1 *Let $\mathcal{S}$ be a subset of a Hilbert space $\mathcal{H}$. The set*

$$\mathcal{S}^* \triangleq \{y \in \mathcal{H} \quad | \quad \langle y, x\rangle \geq 0, \quad \forall x \in \mathcal{S}\}$$

is called the dual *of $\mathcal{S}$.*

Definition 2.2 *A cone $\mathcal{C}$ is called*

- *convex if $\alpha\mathcal{C} + (1-\alpha)\mathcal{C} = \mathcal{C}, \ \forall \ 0 \leq \alpha \leq 1$,*
- *pointed if $\mathcal{C} \cap -\mathcal{C} = \{0\}$,*
- *solid if $\mathcal{C}$ has nonempty interior $\overset{o}{\mathcal{C}} \neq \emptyset$,*
- *self-dual if $\mathcal{C}^* = \mathcal{C}$.*

An important property of a closed convex cone $\mathcal{C}$ which is pointed is that the interior of its dual is given by (see Berman (1973)):

$$\{y \in \mathcal{C}^* \quad | \quad \langle y, x\rangle > 0, \quad \forall x \in \mathcal{C}, x \neq 0\}.$$

Thus the interior of a pointed, closed convex and self-dual cone is also characterized as:

$$\overset{o}{\mathcal{C}} := \{y \in \mathcal{C} \quad | \quad \langle y, x\rangle > 0, \quad \forall x \in \mathcal{C}, \quad x \neq 0\}.$$

In fact, the above description is valid for any closed convex solid cone. This follows from the next result which can be found in Berman (1973).

Lemma 2.1 *Let $\mathcal{C}$ be a closed convex solid cone. Then, its interior is given by:*

$$\overset{o}{\mathcal{C}} = \left\{x \in \mathcal{C} \quad | \quad \langle x, y\rangle > 0, \quad \forall y \in \mathcal{C}^*, \quad y \neq 0\right\}.$$

An immediate consequence of the above lemma is:

Lemma 2.2 *Let $\mathcal{C}$ be a closed convex solid cone. Then, for any element $e \in \overset{o}{\mathcal{C}}$ we have $\mathcal{C} + e \subset \overset{o}{\mathcal{C}}$, and thus $\mathcal{C} + \overset{o}{\mathcal{C}} \subset \overset{o}{\mathcal{C}}$.*

Proof By Lemma 2.1, it suffices to prove that for any $c \in \mathcal{C} + e$, then $\langle c, x\rangle > 0$ for all nonzero $x \in \mathcal{C}^*$. Since by definition $\langle c - e, x\rangle \geq 0$, then applying Lemma 2.1 yields $\langle c, x\rangle \geq \langle e, x\rangle > 0$. □

The following result will be useful in the subsequent proof of the main convergence result.

Lemma 2.3 *Let $\mathcal{C}$ be closed convex cone in a Hilbert space $\mathcal{H}$, such that $\overset{o}{\mathcal{C}} \neq \emptyset$. For any element $e \in \overset{o}{\mathcal{C}}$ we have that $dist(\mathcal{C} + e, (\overset{o}{\mathcal{C}})^c) > 0$.*

Proof Let $e \in \overset{o}{\mathcal{C}}$ and suppose that $\mathrm{dist}(\mathcal{C} + e, (\overset{o}{\mathcal{C}})^c) = 0$. Thus, there exist sequences $x_n \in C$ and $y_n \in (\overset{o}{\mathcal{C}})^c$ such that $\lim_{n\to+\infty} \|e + x_n - y_n\| = 0$. Since $r_n = e + x_n - y_n$ goes to zero and $e \in \overset{o}{\mathcal{C}}$, we have $e - r_n \in \overset{o}{\mathcal{C}}$ for some large n. Therefore, we have $e - r_n + x_n \in \overset{o}{\mathcal{C}}$, as $\mathcal{C} + \overset{o}{\mathcal{C}} \subset \overset{o}{\mathcal{C}}$. Since $y_n \in (\overset{o}{\mathcal{C}})^c$ and $y_n = e - r_n + x_n \in \overset{o}{\mathcal{C}}$ this leads to a contradiction and the proof is complete. □

The following useful property holds for any cone $\mathcal{C}$ with nonempty interior.

Lemma 2.4 *Let $\mathcal{C}$ be a cone with nonempty interior. Then, given any elements $x \in \mathcal{H}$ and $e \in \overset{o}{\mathcal{C}}$ there exists a positive scalar $\alpha > 0$ such that $\alpha e - x \in \mathcal{C}$.*

Proof Let $x \in \mathcal{H}$ and $e \in \overset{o}{\mathcal{C}}$ then there exists $\alpha > 0$ big enough such that $e - \alpha^{-1}x \in \mathcal{C}$. Since $\mathcal{C}$ is a cone, we have $\alpha e - x \in \mathcal{C}$. □

2.3 Conic feasibility problem

We consider the following conic feasibility problem:

$$\text{Find } x \in \overset{o}{\mathcal{C}_1} \cap \mathcal{C}_2, \tag{5}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are given convex closed cones in a Hilbert space $\mathcal{H}$.

As an immediate consequence of Lemma 2.4 we obtain the following.

Theorem 2.2 *Let e be any arbitrary element in the interior of $\mathcal{C}_1$. Then, the conic feasibility problem* (5) *is equivalent to*:

$$\text{Find } x \in (\mathcal{C}_1 + e) \cap \mathcal{C}_2. \tag{6}$$

Proof Let e be in the interior of $\mathcal{C}_1$ and $x \in (\mathcal{C}_1 + e) \cap \mathcal{C}_2$. By using Lemma 2.2 then we have $\mathcal{C}_1 + e \subset \overset{o}{\mathcal{C}_1}$. Therefore, also $x \in \overset{o}{\mathcal{C}_1} \cap \mathcal{C}_2$ holds.

On the other hand, assume that $x \in \overset{o}{\mathcal{C}_1} \cap \mathcal{C}_2$ and let e be any arbitrary element in the interior of $\mathcal{C}_1$. By Lemma 2.4, this implies that there exists $\alpha > 0$ with $\alpha x - e \in \mathcal{C}_1$. Since $\mathcal{C}_2$ is a cone and $x \in \mathcal{C}_2$ we have $\alpha x \in (\mathcal{C}_1 + e) \cap \mathcal{C}_2 \neq \emptyset$. □

We now prove our main result and show that the projection algorithm determines a feasible point in a finite number of iteration steps. In particular, we derive an explicit upper bound on the number of iterations steps involved. Roughly speaking, the number of iterations is proportional to the square of the distance from the starting point to the set of all feasible solutions.

We first need the following preparatory result.

Lemma 2.5 *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be closed convex sets with $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$ and $x_0 \in \mathcal{H}$. Consider the sequence defined by*

$$\begin{aligned} x_1 &= \mathcal{P}_{\mathcal{S}_1}(x_0), \\ x_2 &= \mathcal{P}_{\mathcal{S}_2}(x_1), \\ &\vdots \\ x_{2m} &= \mathcal{P}_{\mathcal{S}_2}(x_{2m-1}), \\ x_{2m+1} &= \mathcal{P}_{\mathcal{S}_1}(x_{2m}), \\ &\vdots \end{aligned} \tag{7}$$

Then, for any $x_0 \in \mathcal{H}$ and any integer m we have

$$\operatorname{dist}(x_0, \mathcal{S}_1 \cap \mathcal{S}_2) \geq \sqrt{\sum_{k=0}^{m} \|x_k - x_{k+1}\|^2}. \tag{8}$$

Proof Let $x_0 \in \mathcal{H}$ and $x \in \mathcal{S}_1 \cap \mathcal{S}_2$. Consider the sequence (x_k) generated as above. For any k we have

$$\begin{aligned} \|x_k - x\|^2 &= \|x_k - x_{k+1} + x_{k+1} - x\|^2 \\ &= \|x_k - x_{k+1}\|^2 + \|x_{k+1} - x\|^2 + 2\langle x_k - x_{k+1}, x_{k+1} - x\rangle. \end{aligned} \tag{9}$$

Since x_{k+1} is the projection of x_k and since $x \in \mathcal{S}_1 \cap \mathcal{S}_2$, we obtain by applying Theorem 2.1 to this projection in the kth step

$$\langle x_k - x_{k+1}, x_{k+1} - x\rangle \geq 0$$

and therefore we obtain

$$\|x_k - x\|^2 \geq \|x_k - x_{k+1}\|^2 + \|x_{k+1} - x\|^2.$$

Now, iterating the inequalities for $k = 0, \ldots, m$ we obtain for any m

$$\|x_0 - x\|^2 \geq \|x_{m+1} - x\|^2 + \sum_{k=0}^{m} \|x_k - x_{k+1}\|^2.$$

Thus

$$\inf_{x \in \mathcal{S}_1 \cap \mathcal{S}_2} \|x_0 - x\|^2 \geq \sum_{k=0}^{m} \|x_k - x_{k+1}\|^2$$

and the proof is complete. □

Recall that in a Hilbert space $\mathcal{H}$ one has to distinguish between two natural concepts of convergence, i.e., weak and strong convergence, respectively. Strong convergence refers to the usual notion of convergence in a normed space. In contrast, a sequence of points $x_k \in \mathcal{H}$ is called weakly convergent to $x \in \mathcal{H}$, if

$$\langle x_k, y\rangle \to \langle x, y\rangle,$$

holds for all $y \in \mathcal{H}$. For finite dimensional Hilbert spaces these concepts are equivalent, but for infinite-dimensional spaces strong convergence of a sequence is a strictly stronger property. Note that the sequence (x_k) converges weakly to an element of

$\mathcal{S}_1 \cap \mathcal{S}_2$. If the stronger condition $\overset{o}{\mathcal{S}_1} \cap \mathcal{S}_2 \neq \emptyset$ holds, then even strong convergence holds (see Gubin et al. 1967). We now prove an amplification of that result for convex cones. Later on we discuss the extension to the general convex case.

Theorem 2.3 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be closed convex cones in a Hilbert space $\mathcal{H}$, possibly of infinite dimension. Assume that $\overset{o}{\mathcal{C}_1} \cap \mathcal{C}_2 \neq \emptyset$. For $x_0 \in \mathcal{H}$ and $e \in \overset{o}{\mathcal{C}_1}$ consider the alternating projection algorithm*

$$\begin{aligned} x_1 &= \mathcal{P}_{\mathcal{C}_1+e}(x_0), \\ x_2 &= \mathcal{P}_{\mathcal{C}_2}(x_1), \\ &\vdots \\ x_{2m} &= \mathcal{P}_{\mathcal{C}_2}(x_{2m-1}), \\ x_{2m+1} &= \mathcal{P}_{\mathcal{C}_1+e}(x_{2m}), \\ &\vdots \end{aligned} \tag{10}$$

Then, we have

- *The sequence (x_n) converges strongly to a feasible point in the intersection $(\mathcal{C}_1 + e) \cap \mathcal{C}_2 \neq \emptyset$.*
- *The algorithm finds a feasible point $\hat{x} \in \overset{o}{\mathcal{C}_1} \cap \mathcal{C}_2$ in at most M iterations with*
$$M \leq \gamma(e)^{-2}\mathrm{dist}(x_0, (\mathcal{C}_1 + e) \cap \mathcal{C}_2))^2, \tag{11}$$
where $\gamma(e) := \mathrm{dist}\big(\mathcal{C}_1 + e, \mathcal{C}_2 \cap (\overset{o}{\mathcal{C}_1})^c\big)$.

Proof Strong convergence has been shown by Gubin et al. (1967). From Lemma 2.3, it follows that $\gamma(e) > 0$. Let k denote the smallest integer such that $\|x_k - x_{k+1}\| < \gamma(e)$. k does exist, since otherwise, by Lemma 2.5, $\mathrm{dist}(x_0, (\mathcal{C}_1+e) \cap \mathcal{C}_2))^2 \geq (m+1)\gamma(e)^2$ for all m, which is impossible. Suppose, e.g., that $x_k \in \mathcal{C}_1 + e$ and therefore $x_{k+1} \in \mathcal{C}_2$. Using Lemma 2.5, if $x_{k+1} \in \overset{o}{\mathcal{C}_1}$, then $\mathrm{dist}(x_0, (\mathcal{C}_1+e) \cap \mathcal{C}_2))^2 \geq k\gamma(e)^2$ and we found a solution in at most $k+1$ steps. Thus we are done. Otherwise $\|x_k - x_{k+1}\| \geq \mathrm{dist}(\big(\mathcal{C}_1 + e, \mathcal{C}_2 \cap (\overset{o}{\mathcal{C}_1})^c\big) = \gamma(e)$, which is impossible. Similarly for $x_k \in \mathcal{C}_2$. Thus the result follows. □

2.4 Convex feasibility problems

We now propose an extension to more general feasibility problems where the associated sets are not necessarily cones. That is we consider the task

$$\text{Find } x \in \overset{o}{\mathcal{S}_1} \cap \mathcal{S}_2, \tag{12}$$

where $\mathcal{S}_1$ and $\mathcal{S}_2$ are given closed convex sets in a Hilbert space $\mathcal{H}$.

Using the standard technique of conification this is easily reformulated as a conic feasibility problem. This then enables us to solve (12) using the previous result.

Conification of an arbitrary subset is defined as follows.

Definition 2.3 *Let $\mathcal{S}$ be a subset in a Hilbert space $\mathcal{H}$. Define* $\mathrm{con}(\mathcal{S}) \subset \mathcal{H} \times \mathbb{R}$ *by*

$$\mathrm{con}(\mathcal{S}) := \{(x, \alpha) | \alpha > 0, \ \ \alpha^{-1}x \in \mathcal{S}\}. \tag{13}$$

The topological closure of $\mathrm{con}(\mathcal{S})$ *in* $\mathcal{H} \times \mathbb{R}$

$$C(\mathcal{S}) := \overline{\mathrm{con}(\mathcal{S})}$$

is called the conification of $\mathcal{S}$.

The following elementary properties are easily established. Here, $\mathbb{R}_+$ denotes the set of positive real numbers.

Lemma 2.6 *The following holds true for any subset $\mathcal{S}$.*

- *If $\mathcal{S}$ is convex, then con$(\mathcal{S})$ is convex.*
- *If $\mathcal{S}$ is closed, then con$(\mathcal{S})$ is relatively closed in $\mathcal{H} \times \mathbb{R}_+$.*
- $\mathrm{con}(\overset{o}{\mathcal{S}}) = \overset{o}{\mathrm{con}}(\mathcal{S}) = \overline{\overset{o}{\mathrm{con}}(\mathcal{S})}$.

Proof The first two claims are easy to prove. For the third one, assume first that (x, α) is an element of $\mathrm{con}(\overset{o}{\mathcal{S}})$. Then $\alpha^{-1}x$ is an interior point of $\mathcal{S}$. Thus, by continuity, there exists $r > 0$ such that for all (y, β) in the r-neighborhood of (x, α) we have: $\beta > 0$ and $\beta^{-1}y \in \mathcal{S}$. This shows the first part. For the second inclusion, assume that (x, α) is an interior point of con$(\mathcal{S})$. Then it is clearly nonzero and for any y in a sufficiently small r-neighborhood of x we have that (y, α) is contained in con$(\mathcal{S})$. This implies that x is an interior point of $\mathcal{S}$ and the first equation has been shown. The second equation follows immediately from the fact that con$(\mathcal{S})$ is relatively closed in $\mathcal{H} \times \mathbb{R}_+$. This completes the proof. □

Obviously the closure of any convex cone is again a convex cone. Therefore the conification $C(\mathcal{S}) = \overline{\mathrm{con}(\mathcal{S})}$ is again a convex cone.

Proposition 2.1 *The convex feasibility problem* (12) *is equivalent to the conic feasibility problem*

$$\overline{(\mathrm{con}(\mathcal{S}_1)} + e) \cap \overline{\mathrm{con}(\mathcal{S}_2)} \neq \emptyset. \tag{14}$$

Here e denotes an arbitrary element in the interior of con$(\mathcal{S}_1)$.

Proof The equivalence of the conditions

$$\overline{(\mathrm{con}(\mathcal{S}_1)} + e) \cap \overline{\mathrm{con}(\mathcal{S}_2)} \neq \emptyset$$

and

$$\overline{\overset{o}{\mathrm{con}}(\mathcal{S}_1)} \bigcap \overline{\mathrm{con}(\mathcal{S}_2)} \neq \emptyset,$$

follows immediately from Theorem 2.2. Since the sets $\mathcal{S}_i$ are closed, we obtain from the relative closedness property of con$(\mathcal{S}_i)$ the equivalence with

$$\overset{o}{\mathrm{con}}(\mathcal{S}_1) \bigcap \mathrm{con}(\mathcal{S}_2) \neq \emptyset.$$

This in turn is equivalent with (12). □

The above result shows how one can approach the general convex feasibility problem (12). First (12) immediately implies $\overset{o}{\mathcal{C}}_1 \cap \mathcal{C}_2 \neq \emptyset$ for the conified sets $\mathcal{C}_i := C(\mathcal{S}_i)$. Thus by applying the projection algorithm to the conified sets we obtain after finitely many steps a feasible point to (14). Then, by de-conifying, one can obtain a solution to (12). Of course, this process works well only, if explicit formulas for the projection operators for the conified sets are available. The open question therefore remains how to express formulas for the projection operators of the conified sets in terms of those for the projection operators of $\mathcal{S}_1, \mathcal{S}_2$. Nevertheless, the process can be carried out in interesting special cases, as we show in the next section.

3 LMI feasibility problem

Constraints involving positive definite (resp., semidefinite) matrices with unknown structure (LMI in the literature) have a wide spread applicability, including, e.g., several problems of interest in control theory (Boyd et al. 1994). For example, consider the following controlled linear system:

$$\frac{dx}{dt} = Ax + Bu. \tag{15}$$

The stabilization problem consists of finding a state-feedback control $u = Kx$ (where K is an unknown matrix) such that the corresponding trajectory goes to zero. It can be shown that the stabilization problem is equivalent to the solvability of the following constraints involving two positive definite matrices with the unknown variables $P = P^T, Y$ (see Boyd et al. 1994)

$$-AP - PA^T - BY - Y^TB^T > 0 \quad \text{and } P > 0.$$

In general, an LMI possesses the following form:

$$M(Y_1, \ldots, Y_k) = \sum_{i=1}^{l} \sum_{j=1}^{k} A_{ij} Y_j B_{ij} + B_{ij}^T Y_j^T A_{ij}^T > 0 \quad (\text{resp.}, \geq 0), \tag{16}$$

where the A_{ij}, B_{ij} are given matrices and the unknown variables are the matrices $Y_1, \ldots, Y_k$. The sign $>$ means that the symmetric matrix $M(Y_1, \ldots, Y_k)$ must be positive definite (all its eigenvalues are strictly positive). In case there exists $Y_1, \ldots, Y_k$ such that $M(Y_1, \ldots, Y_k)$ is positive definite, we say that the LMI is *strictly feasible*. Also, in this case, the domain of the constraints generated by $Y_1, \ldots, Y_k$ has an nonempty interior. When we use the sign $\geq$, this means that the symmetric matrix $M(Y_1, \ldots, Y_k)$ is needed only to be positive semidefinite (its minimal eigenvalue can be zero). In this case, we say that the LMI is *non strictly feasible*(or semidefinite feasibile) which also includes the case that the domain of the constraints generated by $Y_1, \ldots, Y_k$ has an empty interior.

Remark 3.1 Many LMI constraints $M_1 > 0, \ldots, M_p > 0$ are obviously equivalent to one single LMI constraints $M := \mathbf{diag}(M_1, \ldots, M_p) > 0$, where M is nothing else than the matrix formed by the blocks $M_1, \ldots, M_p$.

Strict LMI feasibility problems often appear in different, equivalent formulations (the general form (16) can be expressed in different matrix basis). For example, they may be given either as

$$\begin{aligned} &\text{Find } S > 0 \text{ such that,} \\ &\mathbf{Tr}(F_i S) = b_i, \quad i = 1, \ldots, m, \\ &\text{where the } F_i\text{'s are linearly independent symmetric matrices} \end{aligned} \tag{17}$$

or as

$$\begin{aligned} &\text{Find } x \in \mathbb{R}^m \text{ such that,} \\ &F(x) \equiv F_0 + \sum_{i=1}^{m} x_i F_i > 0, \\ &\text{where the } F_i\text{'s are linearly independent symmetric matrices.} \end{aligned} \tag{18}$$

Such tasks consist in finding a matrix in the intersection of the cone of positive definite matrices and a given affine subspace. In the sequel, we show how to apply our previous procedure to derive a solution to the LMI feasibility problem.

Remark 3.2 Recall that $\mathcal{S}_n, \mathcal{S}_n^+$ denote the sets of real symmetric and positive semi-definite symmetric $n \times n$- matrices, respectively. Thus $\mathcal{S}_n$ is a finite-dimensional Hilbert space, endowed with the standard inner product $\langle X, Y \rangle := \mathbf{Tr}(XY)$. Moreover, the interior $\overset{o}{\mathcal{S}_n^+}$ coincides with the open cone of positive definite matrices.

Applying the preceding conification procedure to these problems leads immediately to the following equivalent formulations to (17), (18)

$$\begin{aligned}&\text{Find } (S,r) \in \overset{o}{\mathcal{S}_n^+} \times \mathbb{R}_+ \text{ such that,}\\ &\mathbf{Tr}\left(\begin{bmatrix} A_i & 0 \\ 0 & -b_i \end{bmatrix}\begin{bmatrix} S & 0 \\ 0 & r \end{bmatrix}\right) = 0, \quad i = 1, \ldots, m.\end{aligned} \tag{19}$$

$$\begin{aligned}&\text{Find } (x_0, x_1, \ldots, x_m) \in \mathbb{R}_+ \times \mathbb{R}^m \text{ such that,}\\ &x_0 \begin{bmatrix} 1 & 0 \\ 0 & F_0 \end{bmatrix} + \sum_{i=1}^{m} x_i \begin{bmatrix} 0 & 0 \\ 0 & F_i \end{bmatrix} > 0.\end{aligned} \tag{20}$$

Of course, problem (19) is a special case of the block-structured conic feasibility problem

$$\begin{aligned}&\text{Find } S \in \overset{o}{\mathcal{S}}{}_n^+ \cap \mathcal{L},\\ &\text{where } \mathcal{L} \text{ is a linear subspace of block-diagonal matrices.}\end{aligned} \tag{21}$$

In particular, problems (19) and (20) are special cases of the conic feasibility problem

$$\begin{aligned}&\text{Find } S \in \overset{o}{\mathcal{D}} \cap \mathcal{L},\\ &\text{where } \mathcal{D} \text{ is a closed convex cone of } \mathcal{S}_n^+,\\ &\text{and } \mathcal{L} \text{ is a linear subspace of } \mathcal{S}_n.\end{aligned} \tag{22}$$

The following result then is a special case of Theorem 2.3.

Theorem 3.1 *Assume that* $\overset{o}{\mathcal{D}} \cap \mathcal{L} \neq \emptyset$. *Then, starting from any element* $S_0 \in \mathcal{S}_n$, *the alternating projection sequence*

$$\begin{aligned} S_1 &= \mathcal{P}_{\mathcal{L}}(S_0),\\ S_2 &= \mathcal{P}_{\mathcal{D}+I}(S_1),\\ &\vdots\\ S_{2m} &= \mathcal{P}_{\mathcal{D}+I}(S_{2m-1}),\\ S_{2m+1} &= \mathcal{P}_{\mathcal{L}}(S_{2m}),\\ &\vdots \end{aligned} \tag{23}$$

finds in a finite number of steps an element $\hat{S} \in \overset{o}{\mathcal{D}} \cap \mathcal{L}$.

For the numerical implementation of the above algorithm it is important that one can explicitly compute the projection operator onto the set $\mathcal{D} + I$. This is difficult in general. We focus on the special case $\mathcal{D} = \mathcal{S}_n^+$, where explicit computations can be made. Thus we consider the LMI feasibility problem

$$\begin{array}{l}\text{Find } S \in \overset{o+}{\mathcal{S}_n} \cap \mathcal{L}, \\ \text{where } \mathcal{L} \text{ is a linear subspace of } \mathcal{S}_n.\end{array} \tag{24}$$

Explicit formulas for the projection operators are then given by the following three results.

Lemma 3.1 *Let $\mathcal{L}$ be a finite dimension linear subspace in a Hilbert space $\mathcal{H}$, with basis vectors $x_1,\ldots,x_p$ of $\mathcal{L}$. Then, the Gramian matrix $G = (\langle x_i, x_j\rangle)_{i,j\leq p}$ is invertible and the metric projection onto $\mathcal{L}$ is given by*

$$\mathcal{P}_{\mathcal{L}}(x) = \sum_{i=1}^{p} \alpha_i x_i, \tag{25}$$

where

$$(\alpha_1,\ldots,\alpha_p)^T = G^{-1}\big(\langle x,x_1\rangle,\ldots,\langle x,x_p\rangle\big)^T.$$

Lemma 3.2 *The metric projection of any $M \in \mathcal{S}_n$ onto $\mathcal{S}_n^+$ is computed as follows. Let $M = VDV^T$ be the eigenvalue-eigenvector decomposition where $D = \mathbf{diag}(d_1,\ldots,d_n)$. Define $\bar{D} = \mathbf{diag}(\bar{d}_1,\ldots,\bar{d}_n)$ with*

$$\begin{array}{ll}\bar{d}_i = d_i & \text{if } d_i \geq 0, \\ \bar{d}_i = 0 & \text{if } d_i < 0,\end{array}$$

then $\mathcal{P}_{\mathcal{S}_n^+}(M) = V\bar{D}V^T$.

Proof By Theorem 2.1, $\mathcal{P}_{\mathcal{S}_n^+}(M)$ is the unique element $\tilde{M}$ such that

$$\mathbf{Tr}[(M-\tilde{M})(S-\tilde{M})] \leq 0, \quad \forall S \in \mathcal{S}_n^+.$$

A simple verification shows that $V\bar{D}V^T$ satisfies the above condition. □

Using the above lemma one can compute the projection onto $\mathcal{S}_n^+ + E$ for any symmetric matrix E. Here a general result is provided.

Lemma 3.3 *Given a closed convex subset S of a Hilbert space $\mathcal{H}$ and any element $x \in \mathcal{H}$. Then, the projection onto $S + x$ is given by*

$$\mathcal{P}_{S+x}(y) = \mathcal{P}_S(y-x) + x, \quad \forall y \in \mathcal{H}\,. \tag{26}$$

Proof Using Theorem 2.1 we have

$$\langle \mathcal{P}_S(y-x) - (y-x), \mathcal{P}_S(y-x) - h\rangle \leq 0, \quad \forall h \in S.$$

This implies

$$\langle (\mathcal{P}_S(y-x)+x) - y, (\mathcal{P}_S(y-x)+x) - (x+h)\rangle \leq 0, \quad \forall h \in S,$$

from which the claim $\mathcal{P}_{S+x}(y) = \mathcal{P}_S(y-x) + x$ follows. □

Based on the preceding lemmas, one can present the alternating projection algorithm explicitly as follows.

Algorithm 3.1 The numerical scheme for solving Problem (24) is given as follows:

- Initialization: Choose any $S_1 \in \mathcal{S}_n$
- Step 1: $S_{2k} = \sum_{i=1}^{p} \alpha_i L_i$, where $L_1, \ldots, L_p$ is basis of $\mathcal{L}$,

$$(\alpha_1, \ldots, \alpha_p)^T = G^{-1}(\mathbf{Tr}(S_{2k-1}L_1), \ldots, \mathbf{Tr}(S_{2k-1}L_p))^T,$$

  and $G = (\mathbf{Tr}(L_i L_j))_{1 \le i,j \le p}$.
  If $\lambda_{\min}(S_{2k}) > 0$ then stop, else go to step 2.
- Step 2: Let $S_{2k} = V\mathbf{diag}(d_1, \ldots, d_n)V^T$ be the eigenvalue-eigenvector decomposition and $\bar{D} = \mathbf{diag}(\bar{d}_1, \ldots, \bar{d}_n)$ given by

$$\begin{array}{ll} \bar{d}_i = d_i & \text{if } d_i \ge 1, \\ \bar{d}_i = 1 & \text{if } d_i < 1, \end{array}$$

  then define $S_{2k+1} = V\bar{D}V^T$.
  If $S_{2k+1} \in \mathcal{L}$ stop, else $k \leftarrow k+1$ and go to step 1.

We now present a more explicit form of our numerical procedure to solve the LMI problem (17) .

Algorithm 3.2 The numerical scheme for solving Problem (17) is given as follows:

- Initialization: Choose any $(S_1, r_1) \in \mathcal{S}_n \times \mathbb{R}$ and define $G := [\mathbf{Tr}(A_i A_j) + b_i b_j]_{i,j \le m}$.
- Step 1: Projection onto the linear space

$$\mathcal{L} = \{(S, r) \in \mathcal{S}_n \times \mathbb{R} | \mathbf{Tr}(A_i S) - b_i r = 0, \ \ i = 1, \ldots, m\}$$

  $(S_{2k}, r_{2k}) = \mathcal{P}_{\mathcal{L}}((S_{2k-1}, r_{2k-1}))$ with

$$S_{2k} = S_{2k-1} - \sum_{i=1}^{m} \alpha_i A_i,$$

$$r_{2k} = r_{2k-1} + \sum_{i=1}^{m} \alpha_i b_i,$$

$$(\alpha_1, \ldots, \alpha_m)^T = G^{-1}[\mathbf{Tr}(A_1 S_{2k-1}) - b_1 r_{2k-1}, \ldots, \mathbf{Tr}(A_m S_{2k-1}) - b_m r_{2k-1}]^T.$$

  If $\lambda_{\min}(S_{2k}) > 0$ and $r_{2k} > 0$ then a feasible solution to the LMI problem (17) is $S = S_{2k}/r_{2k}$ and stop, else go to step 2.
- Step 2: Projection onto $(\mathcal{S}_n^+ + I) \times (\mathbb{R}^+ + 1)$ (by using a translation $e = (I, 1)$):
  $(S_{2k+1}, r_{2k+1}) = \mathcal{P}_{(\mathcal{S}_n^+ + I) \times (\mathbb{R}^+ + 1)}((S_{2k}, r_{2k}))$,
  where
  $r_{2k+1} = \max(1, r_{2k})$,
  $S_{2k+1} = V\mathbf{diag}(\max(d_1, 1), \ldots, \max(d_n, 1))V^T$, and $S_{2k} = V\mathbf{diag}(d_1, \ldots, d_n)V^T$ is the eigenvalue-eigenvector decomposition of S_{2k}.
  If $S_{2k+1} \in \mathcal{L}$, then a feasible solution to the LMI problem (17) is $S = S_{2k+1}/r_{2k+1}$ and stop, else $k \leftarrow k+1$ and go to step 1.

In addition to the above concrete form of the algorithm one can also derive an explicit bound on the number of necessary iteration steps. The next theorem is an immediate consequence of Theorem (2.3) and the following lemma, whose proof is obvious from the formula for the metric projection and the fact that the minimal distance of any positive semidefinite matrix E to $\mathcal{S}_n^+$ is at least (even equal to) $\lambda_{\min}(E)$.

Lemma 3.4 *For any $X \in \mathcal{S}_n$ and $E \in \mathcal{S}_n^+$*

$$\|\mathcal{P}_{\mathcal{S}_n^+ + E}(X) - X\|^2 \geq \lambda_{\min}(E)^2, \quad \textit{if}\, \lambda_{\min}(X) \leq 0.$$

Theorem 3.2 *Assume that $\overset{o}{\mathcal{S}}{}_n^+ \cap \mathcal{L} \neq \emptyset$. Choosing any $E > 0$ and starting from any element $S_0 \in \mathcal{S}_n$, we have that the following sequence*

$$\begin{aligned} S_1 &= \mathcal{P}_{\mathcal{L}}(S_0), \\ S_2 &= \mathcal{P}_{\mathcal{S}_n^+ + E}(S_1), \\ &\vdots \\ S_{2m} &= \mathcal{P}_{\mathcal{S}_n^+ + E}(S_{2m-1}), \\ S_{2m+1} &= \mathcal{P}_{\mathcal{L}}(S_{2m}), \\ &\vdots \end{aligned} \tag{27}$$

converges in a finite number of steps to $\hat{S} \in \overset{o}{\mathcal{S}}{}_n^+ \cap \mathcal{L}$. Moreover, the algorithm finds $\hat{S} \in \overset{o}{\mathcal{S}}{}_n^+ \cap \mathcal{L}$ in M iterations with

$$M \leq \lambda_{\min}(E)^{-2} \mathrm{dist}(S_0, (\mathcal{S}_n^+ + E) \cap \mathcal{L}))^2. \tag{28}$$

In particular, if $E := I$ is chosen as the identity matrix, then

$$M \leq \mathrm{dist}(S_0, (\mathcal{S}_n^+ + I) \cap \mathcal{L}))^2 \tag{29}$$

iterations are sufficient.

4 Extensions

A standing assumption of the previously discussed feasibility problems is that a solution in the interior of one of the constraints ($x \in \overset{o}{\mathcal{C}}_1 \cap \mathcal{C}_2$) is available. When this condition does not hold, the proposed algorithm is not guaranteed to find a solution in finite steps. In fact, although the algorithm is defined for any LMI feasibility problem (strict or nonstrict), convergence in the latter case will only be asymptotic.

One interesting fact about the feasibility problems with constraints having an empty interior, is that one can easily detect the unsolvability of such problems. Indeed, as we show in the sequel, there is always a finite steps procedure for checking *infeasibility*.

4.1 Semidefinite LMI

Consider the semidefinite feasibility problem in the form

$$\begin{aligned} &\text{Find } Z \geq 0 \text{ such that} \\ &\mathbf{Tr}(F_i Z) = b_i, \quad i = 1, \ldots, m, \\ &\text{where the } F_i\text{'s are linearly independent symmetric matrices} \end{aligned} \tag{30}$$

or as

$$\text{Find } x \in \mathbb{R}^m \text{ such that}$$
$$F(x) \equiv F_0 + \sum_{i=1}^{m} x_i F_i \geq 0, \tag{31}$$
where $F_1, \ldots, F_m$ are linearly independent symmetric matrices.

The above formulations consist in finding a solution in the intersection of the cone of positive semidefinite matrices and a given affine subspace. Hence, the MAP procedure can be applied without any conification to provide a solution. However, in contrast to the previous results, we can only guarantee asymptotic convergence.

Define the Gramian matrix G associated to $F_0, \ldots, F_m$ by

$$G = (\mathbf{Tr}(F_i F_j))_{i,j \leq m}.$$

The projection onto the affine space

$$\mathcal{A}_1 = \{Z = Z^T | \mathbf{Tr}(F_i Z) = b_i, \;\; i = 1, \ldots, m\},$$

is given by

$$\mathcal{P}_{\mathcal{A}_1}(Z) = Z - \sum_{i=1}^{m} \alpha_i F_i,$$

$$(\alpha_1, \ldots, \alpha_m)^T = G^{-1}[\mathbf{Tr}(ZF_1) + b_1, \ldots, \mathbf{Tr}(ZF_m) + b_m]^T.$$

Similarly, the projection onto the affine space $\mathcal{A}_2$ associated to (31), where

$$\mathcal{A}_2 = \{F_0 + \sum_{i=1}^{m} x_i F_i | x \in \mathbb{R}^m\}$$

is computed as

$$\mathcal{P}_{\mathcal{A}_2}(F) = \sum_{i=1}^{m} \alpha_i F_i + F_0,$$

$$(\alpha_1, \ldots, \alpha_m)^T = G^{-1}[\mathbf{Tr}((F - F_0)F_1), \ldots, \mathbf{Tr}((F - F_0)F_m)]^T.$$

Based on Theorem 2.3, the following numerical algorithm converges asymptotically (with linear convergence rate) to a feasible point satisfying (30) (resp. (31)).

Algorithm 4.1 *To solve problem* (30) (*resp., problem* (31)), *execute the following steps.*

- *Initialization: Choose any* $S = S^T$ *and let* $S_1 = S$.
- *Step* 1: $S_{2k} = \mathcal{P}_{\mathcal{A}_1}(S_{2k-1})$ (*resp.,* $S_{2k} = \mathcal{P}_{\mathcal{A}_2}(S_{2k-1})$).
  If $\lambda_{\min}(S_{2k}) \geq 0$ *then stop, else go to step* 2.
- *Step* 2: *Let* $S_{2k} = V\mathbf{diag}(d_1, \ldots, d_n)V^T$ *be the eigenvalue-eigenvector decomposition and* $\bar{D} = \mathbf{diag}(\bar{d}_1, \ldots, \bar{d}_n)$ *given by*

$$\begin{cases} \bar{d}_i = d_i & \text{if } d_i \geq 0, \\ \bar{d}_i = 0 & \text{if } d_i < 0, \end{cases}$$

  then define $S_{2k+1} = V\bar{D}V^T$.
  If $S_{2k+1} \in \mathcal{A}_1$ (*resp.,* $S_{2k+1} \in \mathcal{A}_2$) *then stop, else go to step* 1.

4.2 Semidefinite programming

It is a well known consequence of duality theory, that linear programming problems can be recast as convex feasibility problems. The same approach also works for linear optimization over LMI constraints. Explicitly, we show—under the strict feasibility condition—how to obtain in a finite number of steps an approximate optimal solution within a given accuracy.

Definition 4.1 *Given a vector $c = (c_1, \ldots, c_m)^T \in \mathbb{R}^m$ and symmetric matrices $F_0, F_1, \ldots, F_m$, then the following optimization problem*

$$\begin{aligned} &\min c^T x, \\ &\textit{subject to} \quad F(x) := F_0 + \sum_{i=1}^{m} x_i F_i \geq 0 \end{aligned} \tag{32}$$

is called a SDP problem. In addition, the dual SDP problem is defined as

$$\begin{aligned} &\max\{-\mathbf{Tr}(F_0 Z)\}, \\ &\textit{subject to} \quad Z \geq 0, \quad \mathbf{Tr}(ZF_i) = c_i \quad \textit{for } i = 1, \ldots, m. \end{aligned} \tag{33}$$

Using duality theory (see, e.g., Luenberger 1969, Nesterov and Nemerovski 1994, Vandenberghe and Boyd 1996), one can transform a linear cost optimization problem over LMI constraints into a convex feasibility problem. This can be done by just zeroing the duality gap between the primal problem and its dual. Here are the details.

Let p^* denote the optimal value of the SDP (32) as

$$p^* := \inf\{c^T x \mid F(x) \geq 0\},$$

and d^* denote the optimal value of the dual SDP problem (33) as

$$d^* := \sup\{-\mathbf{Tr}(F_0 Z) \mid Z \geq 0, \ \mathbf{Tr}(ZF_i) = c_i, i = 1, \ldots, m\}.$$

The following result is well-known.

Theorem 4.1 *The optimal values of the SDP problem and its dual are such that $p^* \geq d^*$. Moreover, if the primal problem and its dual are both strictly feasible we have $p^* = d^*$.*

Thus, assuming that the primal problem (32) and its dual (33) are both strictly feasible, then the set of optimal solutions is exactly the set of feasible solutions to the LMI in Z, x

$$\begin{aligned} &\mathbf{Tr}(F_0 Z) + c^T x = 0, \\ &\mathbf{Tr}(ZF_i) - c_i = 0, \\ &Z \geq 0, \quad F_0 + \sum_{i=1}^{m} x_i F_i \geq 0. \end{aligned}$$

One can apply the previous algorithm to compute an optimal solution. To see how this goes, let $\epsilon > 0$ and consider the following strict feasibility problem

$$\begin{aligned} &\mathbf{Tr}(F_0 Z) + c^T x = \epsilon, \\ &\mathbf{Tr}(F_i Z) - c_i = 0, \\ &Z > 0, \ \ F_0 + \sum_{i=1}^{m} x_i F_i > 0. \end{aligned} \tag{34}$$

Let x^*, Z^* be optimal solutions to the primal and the dual problems, respectively. Choose any strictly feasible solutions x, Z to the primal and the dual problems and define for $0 < t < 1$

$$x^t = tx^* + (1-t)x \quad \text{and} \quad Z^t = tZ^* + (1-t)Z.$$

By convexity we have $Z^t > 0$, $\mathbf{Tr}(F_i Z^t) - c_i = 0$ and also $F_0 + \sum_{i=1}^m x_i^t F_i > 0$. Choose $\epsilon > 0$ sufficiently small so that

$$t = 1 - \frac{\epsilon}{\mathbf{Tr}(F_0 Z) + c^T x}$$

is between 0 and 1. Note that, by the duality gap inequality, we always have that $\mathbf{Tr}(F_0 Z) + c^T x > 0$, unless we are at the optimum. Since $\mathbf{Tr}(F_0 Z^*) + c^T x^* = 0$ then $\mathbf{Tr}(F_0 Z^t) + c^T x^t = \epsilon$. Therefore, the strict feasibility problem (34) has a solution x^t, Z^t.

On the other hand, if x_ϵ denotes a feasible solution to (34), then x_ϵ provides an ϵ-approximate optimal solution to (32), that is $|p^* - c^T x_\epsilon| \le \epsilon$. Therefore, it suffices to solve (34). To this end, we show how one can cast problem (34) into our framework.

Since the F_i are linearly independent, a solution $F_{-1} := \sum_{i=1}^m \beta_i F_i$ to

$$\mathbf{Tr}(F_i F_{-1}) = c_i \;\; i = 1, \ldots, m,$$

exists and it is given by $(\beta_1, \ldots, \beta_m)^T = G^{-1}(c_1, \ldots, c_m)^T$ (G is the Gramian matrix of the F_i).

Using this matrix F_{-1}, the constraint $\mathbf{Tr}(F_0 Z) + c^T x = \epsilon$ in (34) is expressed equivalently as

$$\mathbf{Tr}(F_0 Z) + \mathbf{Tr}\left(F_{-1}(F_0 + \sum_{i=1}^m x_i F_i)\right) = \epsilon + \mathbf{Tr}(F_{-1} F_0).$$

Now, define $F_{m+1}, \ldots, F_{\frac{n(n+1)}{2}}$ to be a basis of the orthogonal complement of $F_1, \ldots, F_m$ and let

$$X := F_0 + \sum_{i=1}^m x_i F_i.$$

Then, from the identity

$$\mathbf{Tr}(F_i X) = \mathbf{Tr}(F_0 F_i), \quad i = m+1, \ldots, \frac{n(n+1)}{2},$$

we conclude that the constraints (34) are equivalent to

$$\begin{aligned} &\mathbf{Tr}(F_0 Z) + \mathbf{Tr}(F_{-1} X) = \epsilon + \mathbf{Tr}(F_{-1} F_0), \\ &\mathbf{Tr}(F_i Z) = c_i, \quad i = 1, \ldots, m, \\ &\mathbf{Tr}(F_i X) = \mathbf{Tr}(F_0 F_i), \quad i = m+1, \ldots, \tfrac{n(n+1)}{2}, \\ &Z > 0, \quad X > 0. \end{aligned} \tag{35}$$

Therefore (34) is equivalent to the following standard LMI problem

$$\begin{aligned} &\mathbf{Tr}(A_i \,\mathbf{diag}(Z, X)) = b_i, \quad i = 1, \ldots, \tfrac{n(n+1)}{2} + 1, \\ &Z > 0, \quad X > 0. \end{aligned} \tag{36}$$

Here, $A_1, \dots, A_{\frac{n(n+1)}{2}+1}$ are linearly independent symmetric matrices of the form

$$
\begin{aligned}
b_1 &= \epsilon + \mathbf{Tr}(F_{-1}F_0), && \\
b_i &= c_{i-1} && \text{for } i = 2, \dots, m+1, \\
b_i &= \mathbf{Tr}(F_0 F_{i-1}) && \text{for } i = m+2, \dots, \tfrac{n(n+1)}{2}+1, \\
A_1 &= \begin{bmatrix} F_0 & 0 \\ 0 & F_{-1} \end{bmatrix}, && \\
A_i &= \begin{bmatrix} F_{i-1} & 0 \\ 0 & 0 \end{bmatrix} && \text{for } i = 2, \dots, m+1, \\
A_i &= \begin{bmatrix} 0 & 0 \\ 0 & F_{i-1} \end{bmatrix} && \text{for } i = m+2, \dots, \tfrac{n(n+1)}{2}+1.
\end{aligned}
\tag{37}
$$

4.3 Infeasibility problem

Before applying any of the previous algorithms it is of obvious interest to know whether or not a feasible solution exists. Infeasibility, i.e., the nonexistence of a nontrivial solution, can be easily detected by running the algorithm for a dual problem.

Theorem 4.2 *Let $\mathcal{C}$ be a closed convex solid cone and $\mathcal{S}$ be a closed convex set. Then, the following are equivalent*

(i) $\overset{o}{\mathcal{C}} \cap \mathcal{S} \neq \emptyset$.
(ii) $\mathcal{C}^* \cap (-\mathcal{S}^*) = 0$.

Proof The result will be proved by contradiction.
(i) $\Rightarrow$ (ii). Assume that there exists $x \neq 0$ and $x \in \mathcal{C}^* \cap (-\mathcal{S}^*)$. Let y be any element in $\overset{o}{\mathcal{C}} \cap \mathcal{S} \neq \emptyset$. Then, by definition of the dual set and since $-x \in \mathcal{S}^*$, we have necessarily $\langle x, y \rangle \leq 0$. Also, since $y \in \overset{o}{\mathcal{C}}$ then by Lemma 2.1 we have $\langle x, y \rangle > 0$ which leads to a contradiction.
(ii) $\Rightarrow$ (i) If $\overset{o}{\mathcal{C}} \cap \mathcal{S} = \emptyset$, then using the separation theorem for convex sets (see, e.g., Luenberger 1969) there exists a nonzero element $h \in \mathcal{H}$ satisfying

$$
\begin{aligned}
\langle h, c \rangle &\geq 0, \quad \forall c \in \mathcal{C}, \\
\langle h, s \rangle &\leq 0, \quad \forall s \in \mathcal{S},
\end{aligned}
\tag{38}
$$

so that h belongs to $\mathcal{C}^* \cap (-\mathcal{S}^*) = 0$ which leads to a contradiction. □

Let us specialize the above result to the case of semidefinite LMI problems, i.e. to the task of finding a nontrivial solution in the intersection of the cone of positive semidefinite matrices and a linear subspace. First, recall that the dual of a linear subspace $\mathcal{L}$ is its orthogonal $\mathcal{L}^\perp$ and $\mathcal{L} = (\mathcal{L}^\perp)^\perp$. Also, the cone $\mathcal{S}_n^+$ is self-dual: $\mathcal{S}_n^+ = \mathcal{S}_n^{+*}$ (see Berman 1973). Thus from Theorem 4.2 we have the following result.

Corollary 4.1 $\overset{o}{\mathcal{S}_n^+} \cap \mathcal{L}^\perp \neq \emptyset$ *if and only if there exists no nontrivial $P \neq 0$ with $P \in \mathcal{S}_n^+ \cap \mathcal{L}$.*

In particular, the nontrivial solvability of $P \in \mathcal{S}_n^+ \cap \mathcal{L}$ can be decided in finitely many steps, by running Algorithm 3.1 for the conic feasibility problem $\overset{o}{\mathcal{S}_n^+} \cap \mathcal{L}^\perp \neq \emptyset$.

5 Conclusion

A finite step projection algorithm is presented for solving general conic feasibility problems. In particular, the method applies to LMI problems, which in turn, reduce to solve a sequence of symmetric eigenvalue decompositions. The proposed algorithm has several advantages. First, a solution can be obtained in a finite number of steps. It also works for checking certain infeasibility problems. Moreover, the method is well adapted to large scale problems with several small size constraints.

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